ELEMENTARY SCHOOL STUDENTS' INFORMAL AND INTUITIVE CONCEPTIONS OF PROBABILITY AND DISTRIBUTION

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Data and chance are the two related topics that deal with uncertainty. On the discussions of probability and statistics in both research and instruction, the existing literature depicts an artificial separation, to which other researchers (Shaughnessy, 2003; Steinbring, 1991) have already called attention in recognition of the inseparable nature of data and chance. Hence, this paper addresses how to integrate the discussions of distributions and probability, starting from the elementary grades. We report on a study that examines fourth-grade students' informal and intuitive conceptions of probability and distribution through a sequence of tasks for developing their understandings about probability distributions. These tasks include various random situations that students explore with a set of physical chance mechanisms and that can be modeled by a binomial probability distribution.

THEORETICAL FRAMEWORK

Extensive research has been done in probability across a wide range of age groups. In this literature, a body of research has identified and documented different kinds of reasoning under uncertainty and described the development of the probability concept in children (e.g., Fischbein, 1975; Kahneman and Tversky, 1972; Konold *et al*., 1993; Piaget and Inhelder, 1975; Tversky and Kahneman, 1973) and others have focused on how students understand and learn probability and related concepts, such as randomness, ratio and proportion, fairness, and sample space (e.g., Green, 1983; Lehrer, Horvath, and Schauble, 1994; Watson and Moritz, 2003). The earlier studies, in particular by cognitive psychologists Kahneman and Tversky, conducted with college students primarily documented some erroneous conceptions and strategies that students held and employed in judging the likelihood of uncertain events. Such examples include the questions about the likelihood of outcomes and the results of repeated trials in various uncertain or chance situations. Shaughnessy (2003) claims that students' strategies related to these questions have roots in both probability and statistics in nature. Therefore, making meaningful connections between probability and statistics is very important and necessary. Accordingly, Shaughnessy points out a close link between the notion of sample space in probability and the nature of variation in statistics. Furthermore, he suggests that students should be introduced to probability through data. This approach assumes that statistics should motivate probability questions in the context of real data. However, Steinbring (1991) points out that there is a tendency of organizing teaching probability and statistics within a hierarchical structure in a logical sequence, rather than emphasizing the complementarity between the mathematical description and empirical application in stochastic situations. In such a traditional approach, the conceptual link between probability and statistics is not available until the discussion of statistical inference (in advanced levels) in which the idea of probability is imposed.

An important backdrop for this artificial separation in the discussions of probability and statistics can be the historical development of the probability concept. Hacking (1975) notes the duality of the concept of probability during the transformation of the old concept of sign in the medieval periods (i.e., evidence of testimony by the authority found in natural signs) to the inductive evidence (i.e., evidence of things in recognition of the connection between natural signs and frequency of their correctness). This dual nature of probability, which still exists, includes (1) an epistemic notion of probability, referring to support by evidence and (2) a statistical notion of probability, concerning with stable frequencies of occurrences or certain outcomes (Hacking, 1975). The recognition of this duality is essentially important for learning and teaching probability.

An implication of the dual nature of probability is twofold. On the one hand, the epistemic notion of probability emphasizes personal probability relative to our background knowledge and beliefs and thus enables us to represent learning from experience (Hacking, 2001).

On the other hand, the statistical notion of probability underlines stable and law-like regularities in relation to physical and geometrical properties of chance setups and events by calculating relative frequencies from experiments (Hacking, 2001). According to Steinbring (1991), "beginning with very personal judgments about the given random situation, comparing the empirical situation with their intended models will hopefully lead to generalizations, more precise characterizations, and deeper insights" (p. 146). In other words, Steinbring suggests that subjective probabilities based on our knowledge, but not simply a matter of opinion, can be checked by experiment. Kazak and Confrey (2005) point out that when talking about probabilities, one draws upon a variety of evidence, such as personal knowledge or belief, empirical results, and theoretical knowledge. Especially, young students' understandings of probabilities are based on their personal and experiential knowledge about the world. Therefore, the idea of simulation of probability experiments is key to link empirical results to theoretical outcomes.

METHOD

In a response to the calls to make connections between data and chance, our study focuses on the ideas of probability and distribution. We will report on the pilot study conducted with four fourth-grade students (age 9) attending a local elementary school. This study was used to investigate and develop conjectures about the conceptual trajectories along which students' ideas about probability distributions develop, prior to a teaching experiment study. We examined the students' ways of interpreting and reasoning about distributions in natural and chance events through a sequence of tasks. The major sources of data include videotapes of student discussions as they participated in the tasks and their responses in one-on-one interviews, and student-produced artifacts. We report on the qualitative analyses of students' responses and actions that reflect their stochastic reasoning and strategies.

ANALYSIS

This paper discusses how students see distributions in the tasks where they were asked to describe and explain the natural distribution of entities, such as animals in a field and leaves under a tree, and the distributions of objects in designed settings, such as chips dropped on the floor through a tube and marbles dropped in a split-box. Furthermore, it addresses how those understandings evolve into the notion of probability distribution in a particular task where students modeled random rabbit hops by tossing a coin with known probability of ½ to determine where the rabbit would be after 5 hops in repeated trials.

- *Natural distributions*: To examine how students look at various distributions of things in nature, one-on-one interviews were conducted with each student at the beginning of the study. When students were asked to describe what they noticed in the pictures (i.e., the digital photographs of a buffalo herd, sheep herd, and leaves under a tree) and whether they could see a pattern, their responses revealed some statistical aspects as well as causal explanations about the distributions.

Table 1: Characteristics of students' explanations of natural distributions

Variability	"Most sheep gathered up together." "Over here they [buffalos] are kind of spacing out, but over here they look like jamming up a little bit."
Typicality	"Most leaves are under the tree."
Density	"They [sheep] are less and there is like different spots where they eat. More here because they are all together."
Causality	"There are just little that made this far. Probably because the wind would have to be blowing long enough in the right direction for those get there. But that would happen to fewer leaves because mostly they would fall down by the trunk."

In most of the cases, students seemed to view distributions in given natural settings as clumps when they indicated the variability, typicality and density. Moreover, when asked to explain why there were such patterns, students provided causal interpretations rather than explanations indicating any randomness.

- *Distributions of objects*: The following sequence of tasks is intended to support students' understanding of the notion of distribution in designed settings. In these tasks, students conducted various experiments in which they were asked to predict, generate, and interpret distributions of objects.

In pairs, students dropped 20 chips through a tube when the bottom of the tube was 15 inches above the ground. While describing the distribution of the chips, students first focused on where most of the chips were on the floor by showing a hypothetical border around that middle region. For example, Jim said: "Because I was holding this [the tube] right about here. So, they kind of stacked up right here [showing a smaller area in the middle] but they are about around here [showing a larger area around that middle one]." Note that his explanation included the notion of density indicated by the small area right under the tube (i.e., the middle clump) and an expected variability shown by the larger area around that.

When students were asked to conduct the same experiment while holding the tube 30 inches above the ground, their predicted plot of the distribution of the chips still had that middle clump which was a bit bigger, showing the density under the tube and more area outside of the middle one to accommodate the expected spread of the chips (i.e., "this is like a pile and the rest is separate"). After the experiment, students noted the effect of the height variable in the distribution of chips. For instance, they responded: "A lot of them started rolling around," "If it goes up higher, they will spread like almost everywhere," and "Most of them are here and the rest of them are all over."

To follow up students' understanding of the role of chance in the dropping chips activity, they were asked to create a game in which they want to give different points for landing near or far from a target considering the effect of height at which they drop the chips. Each pair chose a higher position than 15 inches to drop the chips in their games considering its effect on the spread (i.e., "We did it at higher level so they roll around more"). In Kate and Tana's game, they divided the sheet into four regions of different sizes as seen in Figure 1. They assigned the highest point to the blue region at the lower right corner as Tana explained, "sometimes the smallest part is hard to get on." In Jim and Brad's game, the bigger circle in the middle had the lowest score while a very small area close to the point where the chips were dropped got the highest point in the game, which they called "the bonus point" (see Figure 1). Jim and Brad also had an idea of "loosing points" that are assigned to the two regions outside of the bigger circle for the chips expected to roll around randomly.

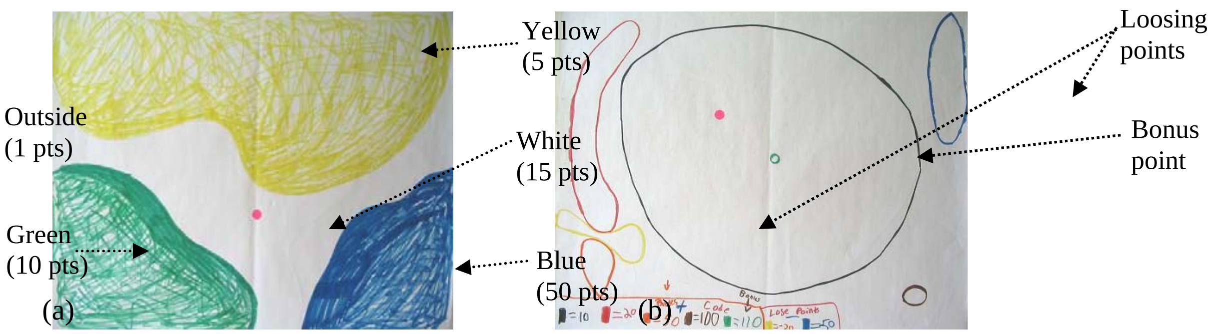


Figure 1: Student-generated games: (a) Kate and Tana's game (b) Jim and Brad's game

To investigate students' understanding of distributions that are generated with the notion of chance inherent in the physical apparatus, students were asked to experiment with an inclined box (see Figure 2) with a centered funnel-like opening on the upper part to drop the marbles and a partition dividing the lower side into two same-size slots (adapted from Piaget and Inhelder, 1975). Students carried out several investigations by letting some marbles fall down through the funnel and watching how they bounced off the middle divider to understand the mechanism of the physical apparatus and possibly to find out an algorithm to predict outcomes. To do so, they dropped different numbers of marbles from each side of the funnel, such as five marbles from each side, or six marbles on the left side and four on the right side, or all on one side. After several experiments, one student said "I think that if we put more on this side [left], it has a bigger chance to go on this side [right] because they are opposites and it might go something like that and in this something like that [showing possible paths from the left-top to the right-bottom and vice versa]." When asked to predict the number of marbles in each side of the split-box, students tended to make their predictions unequal, but close to even, such as 6 to the left and 4 to the right, or 27 to the left and 23 to the right, or 49 to the left and 51 to the right. Although students used the notion of "50-50" to refer to the equal distribution of marbles in each slot when they dropped 100 of them, their predictions for the results were mostly "close to equal" (i.e., "48-52") for 100 marbles. Similarly in Piaget and Inhelder (1975), children at the stage II (7-11 years of age) expected about equal number of balls between the right and left slots, but with no recognition of any equalization as the number of balls increases.

Figure 2: The split-box for marble drops

- *A binomial probability distribution of rabbit hops*: The purpose of this task is to introduce students a situation that can be modeled by a probability distribution and to link the observed frequency of outcomes to the probability of outcomes through a simulation of an uncertain phenomenon. Prerequisite to the task, students were asked to experiment with coin tosses, such as predicting the outcomes of 5 coin-tosses and 10 coin-tosses. Similar to the findings in the previous task, students mostly tended to predict "close-to-even" results (i.e., 6 heads and 4 tails) while some believed in "extremes" (i.e., 1 heads and 9 tails or 10 heads and no tails) thinking that "anything could happen" based on the outcome approach (Konold *et al*., 1993).

In the hopping rabbits activity (adapted from Wilensky (1997) in which one of the subjects created this model in an attempt to make sense of normal distributions), students were asked to predict and then simulate where a rabbit is likely to be after 5 hops in repeated trials by tossing a coin. Students' initial responses revealed a deterministic approach:

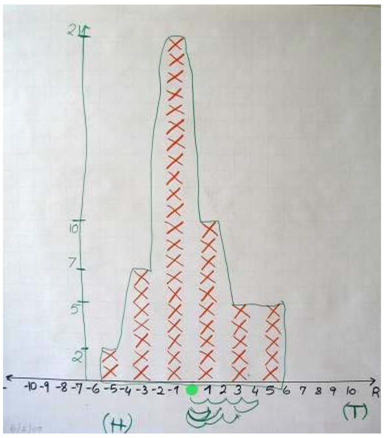
Jim: If I were a rabbit, I'd know where I'd land.
Sibel: You would know?
Jim: Yeah, because I get to do it….Or, I could just tell the rabbit what happens next.

However, introducing the idea of simulation of the rabbit hops with coin tosses (i.e., tails to the right and heads to the left) helped students consider the chance effect on decision-making. Based on the number of hops, students first noted the range of possible outcomes (from -5 to 5 on the number line given that they start at 0). Their responses to "where do you think they are most likely to be after 5 hops?" showed some variation:

Kate: I think most of them on this side [right]
Jim: One. I think it is going to be this.
Brad: Three.
Kate: More here [on four].

Kate's last response related to the most likely outcome after 5 hops opened a new discussion about whether it is possible to land on an even number on the number line after 5 hops. Jim's strategy was to try different combination of five hops to the left and right (see the paths in Figure 3) to convince others about that it is impossible to land on even numbers on the number line after an odd number of hops.

After each group conducted their simulations and plotted their outcomes on the graph paper (Figure 3a), they were asked to interpret them. Their responses involved comparing the individual points ("-1 has the most," "1 is the second") as well as aggregates of data ("There is a majority in the negative side than the positive side"). They also noted that the outcomes were "spaced out" on the graph, which was due to the nature of a discrete random variable that students were asked to model in this task. They acknowledged the likelihoods of different outcomes referring to them as “easy to get,” “hard to get,” and “equally easy/hard to get (or symmetric).” When they attempted to quantify those likelihoods in finding all possible ways to get an outcome (i.e., combinations of 3H and 2T, and a sequence of HTHTH or HHHTT), students made use of different forms of “inscriptions” (Latour, 1990), such as lists, paths, and stacked plots (Figure 3a and b). For instance, Jim’s list of combinations to obtain each outcome in Figure 3b is a critical step that constitutes operative quantification of probabilities when followed by recognizing the respective ordered arrangements of those combinations in a sequence. This example might add a new level of understanding about constructing an idea of chance and probability which, according to Piaget and Inhelder (1975), essentially depends on the ability to use combinatoric operations in random mixture cases.

(a) 

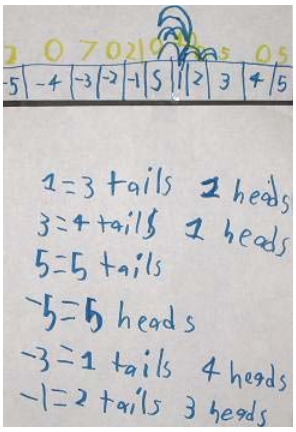
(b) 

Figure 3: Student-generated inscriptions for the rabbit hops.

SUMMARY

The analysis we presented in this paper supports our initial conjecture that students’ spontaneous understanding of distributions in various chance settings could possibly evolve into the notion of a probability distribution which involves a quantitative perception of probabilities. Those spontaneous ideas about distributions involve such notions as aggregates, bunches, middle clumps, spread, and density in natural distributions and distributions of objects in designed settings. Students note patterns in natural distributions and explain them from causally determined viewpoints. In design settings, students tend to implement inquiry methods to examine any systematic pattern and algorithm to explain these patterns. It could be possible that they start believing that chance takes over in the absence of a determining cause in the apparatus. When predicting the outcomes in dropping a number of marbles in the split-box and flipping a coin many times (when the chances are equally likely), students tend to give responses close to equal. This could imply that they believe the outcome would be as close as possible to but not always perfect.

In the simulation of rabbit hops, students could understand these two contexts differently: (1) rabbits can choose to hop right or left and they are equally likely to hop either way and (2) the rabbit hops can be modeled by tossing a coin. According to student responses, the former case encompasses a deterministic view (i.e., “I could just tell the rabbits what happens next”) whereas the latter situation is considered more chance-based (i.e., “Then it’s 50-50”). Once they understand the task as a modeling approach, students begin to describe the likelihood of outcomes qualitatively (i.e., “easy” or “hard to get”) by using various forms of inscriptions. These inscriptions as initial steps in a modeling process suggest that students have intuitive ideas about permutations and combinations which can serve as a strong basis for developing quantitative conceptions of probability and distribution.

In general, the findings from this study suggest that students’ informal conceptions in stochastic topics could be developed into more powerful ways of thinking in probability and statistics through a sequence of tasks. Moreover, the use of simulations to model a natural phenomenon by a binomial probability distribution where students can build upon their previous experiences could be a great potential to link the discussions of probability and statistics. In this modeling approach, it is important to note the role of student-generated inscriptions in supporting their arguments in understanding of distributions in chance events. Furthermore, this study adds to our knowledge relating to children's understanding of the distribution as an overarching idea in the recent studies in statistics while focusing on explaining how children come to understand the important concepts related to probability and distribution in decision making processes.

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