PIAGET'S VIEWPOINT ON THE TEACHING OF PROBABILITY: A BREAKING-OFF WITH THE TRADITIONAL NOTION OF CHANCE?

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Piaget's constructivism and its further developments are used as the conceptual framework to relate, in the learning process, students' age with specific topics in probability and statistics. Such a perspective consists of opposing the notion of chance to that of a reversible sequence and, therefore, to causality. Nevertheless, when the contributions to probability theory developed during the 17th to the 19th centuries are considered, it can be noticed that the concept of chance is a characterization of "our ignorance of the causal chain." This fact motivates two questions which are discussed in this manuscript. The first one consists of understanding what constitutes the breaking-off between Cournot's viewpoint of probability and the traditional one. The second question consists of exploring what kind of probability and statistics teaching would be developed if the traditional viewpoint on chance and probability is considered.

FROM CAUSALITY TO THE NOTION OF CHANCE: PIAGET'S PROGRAM

According to Piaget (1950) and Piaget and Inhelder (1974), children do not have an inborn notion of chance. It is acquired after the acquisition of the corresponding *opposite notions*. This means that chance events can be grasped *only after* identifying a set of phenomena which are not random events. Probability is a secondary notion in the sense that it is a synthesis of two concepts, namely chance and operations: it allows us to provide a structure to the field of fortuitous dispersions. Notions opposite to chance are characterized in terms of the basic structures acquired during the mental development of children. According to Piaget, this development has a central feature: to transit from reversible and composable operations to the irreversible ones.

The analysis of the individual evolution of intelligence addresses the following problem: how the acquisition of logical and mathematical operations depends on concrete actions which in turn are reversible and composable? This development is characterized by three different stages. *First stage* (before 7-8 years old): this period is characterized by the absence of reversible compositions, of nested hierarchies and of totality conservation. At this stage, children can only articulate intuitive relationships. *Second stage* (between 7-8 and 11-12 years old): this period is characterized by the construction of logical operations when clustering and/or counting concrete objects (i.e., objects that can be directly handled). *Third stage* (from 11-12 years old): this period is characterized by formal thinking, namely the ability to link different concrete operational systems applied to different concrete objects, and to translate them in terms of logical implications (propositional logic).

As explained in Beth and Piaget (1961), the structures empirically observed during the mental development of children are related with the irreducible mathematical structures in Bourbaki's sense, namely that a mathematical structure is characterized by its syntax (i.e., a set of rules for combining elements) and that the syntactic properties do not depend on semantic considerations (i.e., concrete meanings of rules and elements). According to Piaget, three types of irreducible structures are empirically observed and so related with specific mathematical structures: firstly, structures such that reversibility corresponds to inversion or cancellation - these ones are similar to algebraic models such groups; second, structures such that the reversibility is the reciprocity - these ones are similar to order structures such lattices; third, structures based on elementary topological homomorphism - these ones are related to spatial structures. Notions opposite to chance are, therefore, based on systems of reversible operations which can be composed between them, namely logical and arithmetical operations. Reversibility and composability allow us to carry out clustering and build groups of objects. However, it can be observed that there are events irreducible to these structures.

This fact leads Piaget to consider *chance* as opposed to two types of causality: firstly, chance is distinct from the mechanistic determinism in which the spatial-temporal connections are ideally reversible. Chance involves, therefore, the idea of non-reversibility. Second, chance suggests some laws, so it is opposed to the notion of miracle: this one is the negation of law. Consequently, both the notion of chance and the intuition of probability constitute derived facts *if compared with the search for order and its causes*. Moreover, Piaget points out that children's *basic* attitude corresponds to the elaboration of causal explanations: the typical child's *why* question essentially represents the search for a reason or a cause.

COURNOT'S CONCEPT OF PHYSICAL CHANCE

Piaget's notion of chance follows Cournot's (1843) interpretation of *physical chance*. From a physical point of view, chance is characterized by irreversible processes, whereas mechanical causality is recognized by reversibility. Chance is, therefore, opposed to causality. So it seems that a notion of causality is acquired after a notion of reversibility. The first stage of children's development is characterized by irreversibility; this explains why in such a stage the notion of chance is non-existent. In other words, there does not exist a mental ability to distinguish reversible events from fortuitous ones.

Cournot's (1843) conception of chance depends on the notion of combination. This last one is suggested by nature, so it is a general and simple concept. Moreover, all scientific syntheses are performed by combining certain fundamental principles or facts. Therefore, logic, algebra, grammar, chemistry, and so on, are based on the theory of combination or combinatoric. The basic idea of combinatorics is to compute the number of possible combinations *without exhibiting each of them.* Therefore, when it is a matter of a random event, what it is relevant to know is the proportion between the number of favorable chances and the number of total chances, where *chance* is understood as a possible outcome (Quetelet, 1853, uses this same sense). Proportion is of interest because it does not change if both of these numbers proportionally increase or decrease. Probability computation includes, therefore, combinatorics, but it contains more. Indeed, it is also applied to situations in which it is not possible to count combinations (for instance, an unlimited number of cases).

Cournot's concern is to apply probability to the real world. In the context of this problem, Cournot characterizes the notion of chance. The fundamental principle regulating human reason when investigating the real world is that each event is produced by a cause. This principle leads Cournot to conceive cause-and-effect chains, the orientation of them being that of time. These chains are, therefore, considered as linear series. In the temporal line, an infinite number of series coexist: they can cross, so the existence of an event is the product of a large number of causal series. This event produces a large number of effects series which remain different and separated.

Cournot distinguishes two types of series: dependent series and series developed in parallel ways (independent series). The events produced by the combination or intersection of phenomena which in turn belong to independent causal series, *are events called fortuitous or random events*.

PROGRESSIVE ACQUISITION OF THE NOTION OF CHANCE

Chance should be considered as the complementary domain of logical composition. Therefore, chance is understood *after* reversible operations are. In this context, probability becomes an assimilation of chance to combinatorial operations: it is not possible to deduce each interference, so all possible combinations should be considered (imagined). *Before* capturing interference of causal series or, at a more concrete level, mixture of mobile objects, children need to build, on the one hand, a system of causal series and, on the other hand, position and movement representations. Briefly, it is a matter of elaborating causality, which is opposed to contingency and chance.

When children at the first stage of mental development are confronted by interference of causal series (as in the physical experiments described in Chapter 1 of Piaget and Inhelder, 1974), they choose one and only one of the following alternatives: (a) interference of causes without acknowledging their independence; (b) independence of causes without acknowledging their interference. This disjunctive choice implies that children avoid grasping the notion of chance events. Intuition about chance begins with the idea of increasing and irreversible mixtures. Nevertheless, random facts can be captured when a large number of them is observed: by so doing, an intentional interpretation is eliminated. Probability and chance are, consequently, notions acquired through the mental development of children, and this explains the relation between mental development stages and the acquisition of combinatorial operations. *First stage*: children do not consider the possibility of a system allowing them to find all the combinations and permutations for some concrete elements. This explains why at this level children do not operate with addition and multiplication. *Second stage*: children understand such systems; they discover them empirically and in an incomplete way. This is due to the fact that the operations they use are applied to empirical objects. *Third stage*: formal thinking allows children to discover some complete combinatorial systems.

Taking these considerations into account, Piaget and his collaborators relate the three stages of mental development with the non-acquisition/acquisition of chance and probability. At the first stage, children do not distinguish *possible* and *necessary*. This is due to the fact that a system with deductive operations is absent. At the second stage, a first development of the notion of chance is found. Children discover deductive necessity (deductive operations). By antithesis with this type of system, children conceive the non-deductive character of isolated fortuitous transformations. By so doing, always at the second stage children arrive at the ability to distinguish necessity and the simpler notion of possibility. On the other hand, complement operations lead them to consider disjunction, which in turn leads to the development of the notion of multiple possibilities (such a fact underlies probability judgments). At this level, the simplest physical expression of chance is conceived as a real mixture of causal series, and not only as a disorder violating an underlying order. This is because children are capable of considering, to a certain degree, the possibility of combinations and permutations of fortuitous arrays. At the third stage: probability judgment is organized in its generality since a synthesis between chance and operations is undertaken. These operations allow children to provide a structure to the field of fortuitous dispersions. In this period, two processes are developed: the construction of combinatorial systems (a combinatorial method is used to consider a mixture of series as a result of transformations, but without order and just performing one particular possibility); the formal thinking allows them to discover proportions: the law of large numbers leads them to conceive the legitimacy of a probabilistic composition of fortuitous modifications in the sense that a proportional dispersion is always regular, and so a rational prevision is conceivable; for details, see Bart (1971) and Ascher (1984).

THE TEACHING OF PROBABILITY AND STATISTICS UNDER PIAGET’S PROGRAM

According to this perspective, the teaching of probability and, by extension, of statistics should consider the following aspects: (a) It should be performed after courses in which the three basic mathematical structures mentioned above are developed. At a university level, this means that Probability and Statistics should be organized after courses dealing with group theory, lattice theory and spatial geometry; (b) Students should be confronted with examples of *physical chance* (which is related with irreversible series). Piaget and Inhelder (1974) offer physical experiments which can be used with children at the second stage of mental development. At an university level, it seems relevant to relate a course of Probability to Experimental Physics emphasizing causality. Simulation of random numbers can also be used in experimental situations in parallel with a course such as Experimental Psychology; see Rabinowitz *et al*. (1989); (c) At a university level, Probability should start with combinatorics. It should be pointed out that combinatorics is a useful tool to count all possible outcomes of a random event emphasizing that such possibilities cannot be causally explained.

A consequence of statement (a) above is that Probability and Statistics are derived from Mathematics in the sense that it is necessary to have an understanding of Mathematics from a Bourbakian point of view, namely to make a distinction between syntaxis and semantics. Piaget and his followers use such a distinction to emphasize an individual’s capability of grasping what is structural (for instance, a reversible operation) from a large number of different concrete or abstract situations. On the contrary, the formalistic school of mathematics uses this distinction to emphasize that the development of all branch of mathematics does not depend on a particular situation. Moreover, Probability Theory has been developed in this way by Kolmogorov (1950). Thus, the teaching of probability and statistics following Piaget’s tradition does not necessarily mean teaching *mathematical* statistics or *theory* of probability, but requires the philosophical and mathematical notions underling probability theory (as explained by Kolmogorov, 1950; see also Gini, 1966).

A consequence of statement (b) above is that Probability and Statistics are viewed as tools for Physics. As a matter of fact, the main point of Piaget's viewpoint concerning the acquisition of the notion of chance is that it is a matter of experimenting with *the physical chance*. Therefore, probability and statistics are related to the physical world. Thus, probability and statistics not only deal with real data, but also become the tools to describe physical chance.

CHANCE, OUR IGNORANCE ON THE CAUSAL CHAIN

The relationship between probability, statistics and physics is not new. It can be traced in Huygens (1656), Arbuthnot (1710), Price and Bayes (1763), Laplace (1774) and Quetelet (1853). Nevertheless, the relationship between *chance* and *causality* seem not only to be different from that established by Cournot and Piaget, but also to be related to specific philosophical and theological aspects, which in turn were elaborated from Newton's, Bentley's and Boyle's perspective; for details, see San Martín (2005). The semantic field of the notion of *chance* can be grasped from the arguments claimed in order to publish Bayes' (1763) essay at the *Philosophical Transaction*. These arguments are contained in Price's letter addressed to John Canton (Bayes and Price, 1763), who in turn published in the *Philosophical Transaction* around twelve manuscripts on experimental philosophy (i.e., experimental physics); for details, see San Martín (2005).

First argument: why was a letter to Canton addressed for deciding the publication of Bayes' essay? Experimental philosophy is closely interested in the subject developed in Bayes' essay since both are concerned with the establishment of eventual causes from actual effects. This agreement can indeed be understood from Canton, an experimental philosopher.

Second argument: why should experimental philosophy pay attention to Bayes' essay? As pointed out above, the concern of experimental philosophy is to look for eventual causes from actual effects. The orientation of causal series is the same as time, so the problem reduces to making inferences about future events from past events: the key idea is to use available information (i.e., former instances) to deduce the likely future consequences. Moreover, the larger number of experiments we have to support a conclusion, the more reason we have to take it for granted (Bayes and Price, 1763). Nevertheless, what is relevant is to determine "in what degree repeated experiments confirm a conclusion […] which, therefore, is necessary to be considered by any one who would give a clear account of the strength of *analogical* or *inductive reasoning*" (Bayes and Price, 1763). Bayes' rule (i.e., the rule of inverse probability) is, therefore, considered as a tool for ensuring this type of inductive inference. The use of probability as a tool for quantifying *likelihood degrees* in inductive reasoning is traditional in the sense that can already be traced in Huygens' (1673, 1690) inductive reasoning in Physics, the degrees of likelihood being previously developed in Huygens (1656); for details, see San Martín (2005).

Third argument: why do causal series underly inductive inference? To evaluate the relevance of Bayes' contribution in the context of experimental philosophy, Price considers it as the complement of De Moivre's contribution, namely the Law of Large Numbers, the concern being to find the probability of the occurrence of an event (i.e., the probability of an effect) from a great number of trials (i.e., causes). Bayes' contribution leads to a solution of the inverse problem (i.e., from effects to causes), namely "the number of times an unknown event has happened and failed being given, to find the chance that the probability of its happening should lie somewhere between any two named degrees of probability" (Bayes and Price, 1763). Both De Moivre's solution and Bayes' solution are understood assuming as a principle that a thing cannot occur without a cause which produces it. This is correctly considered as a *principle* because causal series are viewed as the actual government of a Deity: "to shew what reason we have for believing that there are in the constitution of things fixt laws according to which events happen, and that, therefore, the frame of the world must be the effect of the wisdom and power of an intelligent cause; and thus to confirm the argument taken from final causes for the existence of the Deity" (Bayes and Price, 1763). Bayes' solution should be understood in this quote: "It will be easy to see that the converse problem solved in this essay is more directly applicable to this purpose; for it shew us, with distinctness and precision, in every case of any particular order or recurrency of events, what reason there is to think that such recurrency or order is derived from stable causes or regulations in nature, and not from any irregularities of chance" (Bayes and Price, 1763).

These considerations can directly be traced in De Moivre (1718) as Price himself pointed out; for details, see San Martín (2005). Moreover, De Moivre exemplifies these considerations with Arbuthnot's (1710) contribution. Arbuthnot considers that the *physical balance* that is maintained between the number of men and women is a footstep "of the Divine Providence to be found in the Works of Nature." This fact is "not the Effect of Chance but Divine Providence, working for a good End." In other words, Chance is opposed to Divine Providence, *but when considering Divine Providence as the government of a Deity through causal laws.* Since these natural laws have a precise objective, in the particular case Arbuthnot is considering that "Species may never fail, nor perish," a right moral behavior is precisely contrary to the attempt of avoiding, in the context of human relationships, that a natural law works. It should therefore be mentioned that De Moivre's conception of the role of natural laws nearly follows that of Arbuthnot (this type of discussion was developed in puritan circles; see Ames, 1643).

In the preface of Huygens' (1692) English translation, Arbuthnot explicitly considers Huygens' method in the context of physics, the advantage being that the computation of a likelihood degree means avoiding a computation of mechanical aspects underlying the observed event. In this context, *chance* is conceiving as *the ignorance of an exact cause producing an even and, therefore, an explicit confession that events are due to causal laws*. Moreover, he insists that this is applicable to both "the great Events of the World, as in ordinary Games," the difference being just the complexity of computations.

The opposition between *chance* and *design (art)* in the sense that all things occur due to a cause, and the idea that causal laws are the expression of a *Deity* (namely, a God who actually governs all physical and social events) are explicitly found in Newton's conception on natural laws, as De Moivre himself acknowledged; for details, see San Martín (2005). Moreover, Laplace's philosophy on chance and probability essentially develops the traditional one already summarized above; for details, see, e.g., Laplace (1774) and San Martín (2005).

TEACHING PROBABILITY AND STATISTICS UNDER THE TRADITIONAL PROGRAM

The traditional notion of chance is, therefore, related to that of inductive inference. As a matter of fact, all events, either social, moral or physical, occur with a cause. The problem consists precisely in detecting *a particular* cause among a set of possible causes which generates the event of interest. The experience of related phenomena provides data to induce such a particular cause, but the complexity of the causal chain, along with the (historical) limitation of human knowledge, lead to performance of a *probabilistic* induction (induction was always a probabilistic induction for Price, Bayes, De Moivre, Laplace, Quetelet, etc.). Thus, chance is an expression of our ignorance and, in the context of inductive reasoning, probability is the more sure complement of human knowledge. Therefore, the teaching of Statistics under the traditional approach leads to placing the accent on epistemological considerations in the sense that when analyzing complex systems under an inductive inferential viewpoint, it is necessary to measure possible inductive errors. This means that knowledge is not definitive and that experience must be accumulated to continually revise inductive procedures.

Under this perspective, the teaching of Statistics and Probability should be developed in parallel with Philosophy and eventually with Theology. In Philosophy, the subject to be discussed is the relation between empiricism and inductive inference. Illustrations are needed to show how inductive inference can be incomplete and erroneous. In Theology, the relationships between natural theology (mainly developed from puritan theology) and the rise of experimental philosophy should be explained; see Hooykaas (2000). Consequently, Statistics and Probability need to be introduced as *epistemological disciplines* helping to formalize inductive inference in all discipline: Physics, Social Sciences, Biology, etc. This tradition would explain why Statistics is widely used in these disciplines; see also Gini (1966). From this traditional perspective, Mathematics is *not* the main related discipline since a differentiation between *probability* and *measure of a probability* is made; see, e.g., Laplace (1774): Mathematics is related to the second aspect, whereas the first one is related to a procedure of reasoning which can be applied even in fields in which computation is eventually impossible (due to its complexity). Thus, the teaching of Statistics and Probability *precede* the teaching of Mathematics and could be in parallel with Physics.

FINAL REMARK

The teaching of Statistics and Probability involve, in a certain sense, the teaching of specific philosophical views of nature, causality and epistemology. Once *a* viewpoint is chosen (the ones developed in this note), Statistics becomes either a transversal discipline (because it is viewed as an epistemological branch) or a complementary one (because it is viewed as the complement of Physics and Mathematics).

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