CHANCE AND NECESSITY: THE LANGUAGES OF PROBABILITY AND MATHEMATICS

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The idea of viewing mathematics as a language of communication has been espoused for many years. In England, this was the starting point for the influential report on the teaching of mathematics by Cockcroft. A much earlier reference is that of Tobias Dantzig in his seminal book, Number–The Language of Science; this would now perhaps be called, Mathematics–The Language of Scientific and Social Disciplines. In some ways, probability can be seen part of mathematics and would be a strong sub-discipline which drives and underpins much work in the social sciences, as well as in science such as quantum mechanics and genetics. The original subsets of mathematics were number, algebra, and geometry, though there are many other fields of mathematics now; it can be argued that probability is quite distinctive, as is statistics. This paper will explore the similarities and differences between the languages of probability and mathematics.

INTRODUCTION

The title of this paper is inspired by Jacques Monod (1971) who discussed genetics and the influence of chance on mutation and change. The original models of the world put God at the centre of the universe as the creator and guardian of evolution. This is still espoused by a significant minority across the world who have invented the new and seemingly more intellectual title of Intelligent Design as a means of gaining credibility. It was aimed to restore faith in God and rule chance out of our lives. However, Monod notes that “Chance alone is at the source of every innovation, of all creation in the biosphere. Pure chance, only chance, absolute but blind liberty is at the root of the prodigious edifice that is evolution. […] It is today the sole conceivable hypothesis, the only one that squares with observed and tested fact.”

Of course the reality is, and has always been, thus (rather than “intelligent design”) in daily life with many events of chance and decisions which need to be taken on the basis of estimating likelihoods of various events. Nevertheless we hanker for the order underpinned by (blind) faith.

Early in man’s history and still the case for other animals today, is the daily search for sustenance and shelter, and what may be necessary to meet the daily needs of life: there is always a chance that these needs are not met, though civilisation and progress has reduced this possibility substantially, at least for humans. Hence, since these early times, there has been a core belief in God and the need for sacrifices either to please him or to supplicate him to be more bountiful. This element of chance has always been present in the history of life. It is a rather emotive link: what does or does not happen matters, and – being creatures of habit, we want to minimise such uncertainty.

For the rest of the paper, we will use the word ‘uncertainty’ to encompass chance and its emotive connotations; we will use the word ‘probability’ for more formal approaches, devoid of emotion; chance will be used as the intermediary between these two inter-connected but distinct ideas. It is well known that models for chance did not develop into the ideas of probability for many centuries, well after many other examples of mathematical thinking.

The need for counting and geometrical ideas evolved much earlier. The development of these ideas can be seen in many cultures from many centuries before the Common Era (BC). Indeed a whole range of counting strategies have been documented such as in Zaslavsky (1999). Subsequently geometrical ideas were also developed and catalogued in what is regarded as one of the first books to encompass and present mathematics as a systematic and rigorous study – the Elements of Euclid, of which there have been many editions, both in the original language and then translated into virtually all the languages of instruction across the world. Hence mathematical thinking can be dated to have begun over two thousand years ago; yet probabilistic thinking has developed only a few hundred years ago.

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MATHEMATICAL THINKING

In some ways mathematical thinking is a refuge from the world, especially when the Platonic approach to mathematics is adopted. Into his academy, he proclaimed "Let no man who does not know mathematics enter". At the time the pinnacle of mathematics was Euclid's Elements, proving, beyond doubt, the absolute truth of geometry – based at it was, on 'incontrovertible' definitions, axioms and postulates. Of course these included the dubious fifth postulate about parallel lines which led to ultimately fruitless attempts to prove it. Many, such as Saccheri spent a lifetime on the task (one of the biggest near-misses in history) but it was left to Lobachevsky and Riemann in the 19th century to eventually develop non-Euclidean geometries, which, paradoxically, are now seen as more accurate models for the universe.

Nevertheless, the heart of Greek geometry remains and mathematics is seen as the search for theorems derived from shared definitions and axioms. It was Hilbert who spoke about de-contextualising definitions from any meaning and hence not using possible intuitions about ideas in proofs. So proof became the distinguishing feature of mathematics from all other disciplines, and also the aim for other disciplines to aspire to. One only has to look at Kant and other philosophers who hankered for such an approach in their own discipline.

It was many centuries later that a similar formalisation was developed for number systems. In some ways, the ideas of number were too basic for the need of formalisation to be recognised. Even now, the formalisation of number by Peano's axioms is seen as rather complex and does not even form part of mathematics courses in many universities. Some university mathematics courses do try to follow the dream of Bourbaki that all mathematics should be developed formally and rigorously. The constraints deriving from Russell's paradox and Gödel's theorems are all too real – as are the limitations of such abstraction for virtually all students.

Proof remains as a key distinguishing feature of mathematical thinking. It allows the development of ideas according to an agreed format which can be checked and analysed across the world and independent of reality. It is an approach which transcends the normal language of communication, be that English, Chinese, Japanese, French, German or Spanish. In education, mathematics is still portrayed as a language of communication.

Another aspect of mathematical thinking is its relationship to our perceptions of the world. Sometimes, the ideas of mathematics conform to our normal perceptions, such as in counting or in geometrical shapes. These ideas fit and build on our intuitions and models for the world. Sometimes, however, mathematical ideas are developed which do not fit neatly and can form hurdles for understanding.

Two such examples are discussed here as exemplary, but there are many more. It should also be noted that these examples are normally taught to older children after at least five years of education, which contrasts with probabilistic ideas which are considered in the next section. The first example in mathematics is the existence of negative numbers and the second example is the derivation of complex numbers. Children always learn the counting numbers first and often have problems when the notion of zero is introduced. The need for zero only arose because of the difficulty of calculation in number systems without place value. Children find the idea of zero difficult initially but soon adapt with the models presented in school.

Negative numbers are developed from an abstract need to extend mathematics and thereby model many situations: the formation of the ring of integers provides a surprisingly powerful model. In some ways negative numbers are counter-intuitive initially and stories of owing/borrowing or up/down are told as ways to introduce an essentially artificial construction. The product of negative numbers is a rather complex hurdle which many students never really master, except by rote memory. The wide range of applications of integers justifies the abstraction. This would normally be undertaken for secondary age students, around grade 6.

Much later, another intuition has to be challenged: that the product of a number by itself is always positive. Again this is an abstraction demanded by mathematics; the surprise is that it is very useful in various models, some of which still relate to reality, even though the notion of complex numbers is abstract. The introduction of complex numbers in grade 10 or above is counter-intuitive and often not accepted or understood by much of the population.

These two examples show another feature of mathematical thinking in that it can require new models to understand seemingly counter-intuitive ideas. In some ways this could also be thought of as a lack of intuition rather than being against normal intuition. Another key feature is that there is no link to emotion in the introduction of such ideas in school. While students may struggle with ideas of negative or complex numbers, and suffer stress in understanding the concepts, there is no underlying emotion relating to these abstractions. This is rather different to ideas relating to uncertainty where emotions can and do intrude.

PROBABILISTIC THINKING

Probability, as a separate discipline emerged rather late in history: most put its origins to correspondence between Fermat and Pascal in resolving gambling problems set by the Chevalier de Mere, who was concerned on settling a fair return relating to games of chance in the 17th century. The key texts are David and Hacking: their respective titles of “Games, Gods and Gambling” and “The Taming of Chance” evoke the historical context effectively.

Yet, as noted above, uncertainty has been part of life since eternity–literally so, if the genetic scientists are to be believed in their belief in chance mutations. Yet man found it hard to conceptualise pertinent ideas to “tame” it and move away from the influence of “God” for many centuries after ideas of mathematics had been developed in geometry by the Greeks. This is not the place to discuss why it took so long–the two references chronicle the events in much depth and detail. Nevertheless, it is important to remember this relatively slow evolution of ideas when planning the curriculum for probability. Of course, the Greeks had a rather idealistic view of the world where thought was rated much higher than mere empirical facts. Probability could not, in such a conception be seen as linked to repeated experiments.

Chance is part of life but did not prove easy to formalise; in some ways, even the core idea of probability is hard to formalise. One only needs to witness the deep intellectual arguments between the frequentist and the subjective notion of probability. For any particular situation, it is not clear why one should assign either equal or different probabilities to the underlying events. It is here that emotion can impinge quite strongly. Not only may the evidence be equivocal but one may also associate a certain personal viewpoint, partly in rooting for a particular event to happen. That is the basis and essence of all gambling: at least two people must take a different view on the chance of a specific outcome for betting to be viable. Perhaps that is one reason that gambling is banned in some cultures such as Muslim.

Equally it is only very speculative that relative frequencies from the past should continue in the future. There is no absolute reason that patterns from the past should repeat in the future. A famous example is that a sequence of alternate heads and tails for hundreds of tosses should lead one to doubt the validity of the reported experiment–a version of this story begins the famous play by Tom Stoppard about Rosencrantz and Guildenstern. It is not clear that such an approach of using patterns from the past is beneficial. Most important is that one has to look at an event from an individual rather than a group perspective. So, even if the probability of an event may be low according to previously collected data, the impact of the event differs for each individual and, there is a ‘why me?’ query implicit in any evaluation of uncertainty.

The first axiomatic approach to probability was only in the 20th century by Kolmogorov. The text is, as one would expect, very formal and precise, and relatively complicated and hard to understand for non-mathematicians. If taught in university, the subject may well be presented within the context of measure theory, which is only needed for mathematical credibility, adding no extra meaning in the process. This may be relevant to discuss more widely, but the focus here is on the school curriculum.

However, Kolmogorov, in common with axiomatic approaches to mathematics, gives no clue to the meaning of probability. For mathematics, it must be thus but this is insufficient for students and children learning these ideas in school or university. There are two main approaches. One is experimental and based on previous frequencies of a phenomenon being studied. The other approach is a subjective approach, based on a range of previous experience, including–but not exclusively–frequencies. The former approach, with its own inherent problems is usually adopted as the initial approach in school, not least because it can lead to numbers being applied and then manipulated. The latter subjective approach is fraught with problems and, as the name implies, its subjective nature leads to different answers by individuals. Yet it also preserves the emotive elements of uncertainty.

The frequentist approach is not devoid of problems, as is apparent in the way individuals interpret probabilistic assumptions, as became apparent in the recent financial meltdown in 2008. The common issue, rarely discussed in depth, is why one might think that a pattern from the past may be replicated in the future, particularly when variation is a key feature. This is very different to the scientific paradigm of looking for causes and effects. Indeed the scientific approach fits well with theology if one ascribes God as the first cause. It is for this reason that God features in the title of the book by David.

The subjective approach is even more fraught with problems and may well not lead to numbers, but perhaps to an ordinal scale which is not easy to manipulate mathematically. It is even worse that it leads to different answers by individuals and so is therefore hard to develop in a school situation. Mathematics, since the time of the Greeks, has been imbued with notions of truth and certainty. It is not easy for mathematics teachers to deal with less precise ideas.

There are many counter-intuitive notions in probability; unlike mathematics, these arise immediately one starts studying the subject, with limited opportunities for understanding why these are misconceptions. The ideas are emotive and emotionally powerful and so hard to shift, even with the authority of a teacher or of mathematics. One may accept the ideas superficially but stick to different beliefs internally. For one has lived with uncertainty since birth and developed some powerful intuitive ideas. In school, the idea of probability may immediately conflict with these fundamental intuitions.

Subsequent ideas can also conflict with intuitions and wrong heuristics remain universal amongst adults, as shown in the ground-breaking work of Tversky and Kahneman. We deal with just two such ideas here but there are many more. Young children believe that a six is harder to roll on a die, simply because many board games require it as a starting number; adults also believe that one usually has to wait for a bus longer than half the time interval between them. Of course, both are right, in some ways, as a means of thinking about waiting times rather than probabilities. These are both examples of the availability heuristic. Another heuristic relates to representativeness; here it is thought that a tail is more likely after five consecutive heads. There is much research in this and hence not discussed further here.

The underlying notions concern fairness and randomness. However, the key idea is that there are a range of counter-intuitive notions, linked to emotional reactions, for children whilst they learn about chance. Overall, we are not always very reflective or objective in our thinking. Sometimes we may think that a glass is half-full; sometimes that it is half-empty. Mathematically the situations are identical: but the language has a quite different significance.

SCHOOL MATHEMATICS

School curricula need to encompass the ideas of mathematics, as well as probability discussed above. Mathematics is also taught to all pupils and so needs to capture their interest and hence the increasing importance given to incorporating applications of mathematics into the curriculum. In this case, probability certainly lends itself to real-life examples. Unfortunately this is too often not the case, with use of urns and bags as pseudo-real examples, where the context is soon forgotten when the ideas of probability are developed.

The core idea of mathematics as a language of communication, as espoused in the Cockcroft report (1982) is important and applies equally to probability. Both disciplines need to be seen as ways or models to view the world. Mathematics and probability give means to make sense of reality. This is studied in greater depth by Tobias Dantzig in his seminal book, Number–The Language of Science. This would now be a book perhaps called, Mathematics–The Language of Scientific and Social Disciplines, and include probability and statistics as key disciplines. The original subsets of mathematics were number, algebra, and geometry, though there are many other fields of mathematics now; it can be argued that probability is quite distinctive, as is statistics.

Mathematical thinking is precise and logical. Probability can also be presented in that way, in terms of its basic rules; yet the formulation of chance itself has complications. For mathematics, the basic ideas of number and geometry are in accord with everyday experience. However, as noted above, even the frequentist approach to the meaning of uncertainty can be counter-intuitive. This is a significant difference to which educationalists need to pay attention. There are counter-intuitive notions in mathematics, but these arise relatively later in the study of the discipline and from abstractions within the discipline. In probability, counter-intuitive paradoxes abound, even in the relatively early stages of the subject. This needs to be borne in mind firmly by educationalists.

CONCLUSIONS

We have discussed mathematics as a language of communication and argued that probability should also be viewed in a similar way. Probability is part of mathematics and is a strong sub-discipline which drives and underpins much work in the social sciences, as well as science such as quantum mechanics and genetics. This paper has explored the similarities and differences between the languages of probability and mathematics. Precision and logic underpin both, but the different nature of underlying intuitions has a strong influence in how the two sorts of thinking should be presented in school.

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