2003 Exam
Example Solutions

1. (a) The ACF is defined to be

$$\begin{aligned}\rho(u) &= \mathrm{cor}[Y_{t+u}, Y_t] \\ &= \mathrm{cov}[Y_{t+u}, Y_t]/\mathrm{var}[Y_t] \\ &= \gamma(u)/\gamma(0)\end{aligned}$$

where γ is the autocovariance function of Y_t. This is well defined because the correlation is independent of t.

(b) Proceed by cases:

$$\begin{aligned}\gamma(0) &= \mathrm{var}[\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}] \\ &= (1 + \theta_1^2 + \theta_2^2)\sigma^2\end{aligned}$$

$$\begin{aligned}\gamma(1) &= \mathrm{cov}[\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}, \varepsilon_{t-1} + \theta_1\varepsilon_{t-2} + \theta_2\varepsilon_{t-3}] \\ &= (\theta_1 + \theta_1\theta_2)\sigma^2\end{aligned}$$

$$\begin{aligned}\gamma(2) &= \mathrm{cov}[\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}, \varepsilon_{t-2} + \theta_1\varepsilon_{t-3} + \theta_2\varepsilon_{t-4}] \\ &= \theta_2\sigma^2\end{aligned}$$

and $\gamma(u) = 0$ for $u > 2$.
Thus the autocorrelation function is:

$$\rho(u) = \begin{cases} 1 & u = 0, \\ \dfrac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} & |u| = 1, \\ \dfrac{\theta_2}{1 + \theta_1^2 + \theta_2^2} & |u| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Start with the equation

$$Y_{t+u} = \phi Y_{t+u-1} + \varepsilon_{t+u-1}.$$

Now multiply through by Y_t,

$$Y_{t+u}Y_t = \phi Y_{t+u-1}Y_t + \varepsilon_{t+u-1}Y_t$$

and take expectations to obtain:

$$\gamma(u) = \phi\gamma(u-1).$$

This recursion also applies to the autocorrelation function

$$\rho(u) = \phi\rho(u)$$

and we know that $\rho(0) = 1$. Putting all this together we obtain:

$$\rho(u) = \phi^{|u|}$$

for $u = 0, \pm 1, \pm 2, \ldots$.

(d) Again, this uses the standard trick.

(i) Start with

$$Y_{t+k} = 0.8Y_{t+k-1} + \varepsilon_{t+k} + 0.7\varepsilon_{t+k-1} + 0.6\varepsilon_{t+k-2},$$

multiply through by Y_t and take expectations to obtain

$$\gamma(k) = 0.8\gamma(k-1) \qquad \text{for } k \geq 3,$$

and hence

$$\rho(k) = 0.8\rho(k-1) \qquad \text{for } k \geq 3.$$

(ii) Start with

$$Y_{t+2} = 0.8Y_{t+1} + \varepsilon_{t+2} + 0.7\varepsilon_{t+1} + 0.6\varepsilon_t,$$

multiply through by Y_t and take expectations to obtain

$$\gamma(2) = 0.8\gamma(1) + 0.6\sigma_\varepsilon^2,$$

and hence

$$\rho(2) = 0.8\rho(1) + 0.6\sigma_\varepsilon^2/\gamma(0).$$

(e) Y_t can be approximated by the series defined by

$$Y_t = \phi Y_{t-1} + \varepsilon_t,$$

where ϕ is just slightly less than 1. The estimated ACFs for the two series should be very similar, so the estimated ACF for the original series should show very slow exponential decay.

2. (a) A series Y_t is (strictly) stationary if for any $k > 0$ and $u > 0$, the distribution of

$$(Y_{t_1}, \ldots, Y_{t_k})$$

is the same as the distribution of

$$(Y_{t_1+u}, \ldots, Y_{t_k+u}).$$

A time series is mixing if values which are far apart in time are near-independent. One particular mixing condition is:

$$\sum_{u=-\infty}^{\infty} |c_{XX}(u)| < \infty$$

(b) The power spectrum is defnied by:

$$f_{XX}(\lambda) = \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} c_{XX}(u) e^{i\lambda U},$$

and describes how much variability is present in the series at frequency λ. (This is meaningful because of the Cramér representation.)

(c) The autocovariance function of white noise is:

$$c_{XX}(u) = \begin{cases} \sigma_\varepsilon^2 & u = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting this into the definition of the power spectrum, we obtain

$$f_{XX}(\lambda) = \frac{1}{2\pi} = \sum_{u=-\infty}^{\infty} c_{XX}(u) e^{i\lambda U} = \frac{\sigma_\varepsilon^2 e^{-i\lambda 0}}{2\pi} = \frac{\sigma_\varepsilon^2}{2\pi}$$

(d) $\mathcal{A}$ is linear and time invariant if

$$\mathcal{A}[\alpha X + \beta Y](t) = \alpha \mathcal{A}[X](t) + \beta \mathcal{A}[Y](t)$$

and

$$\mathcal{A}[L^u X](t) = L^u \mathcal{A}[X](t)$$

where L is the lag operator defined by $LX(t) = X(t-1)$.

(e) The transfer function is

$$A(\lambda) = \sum_{u=-\infty}^{\infty} a(u) e^{-i\lambda u}$$

(f) The transfre function is

$$A(\lambda) = 1e^{-i\lambda 0} - 1e^{-i\lambda s} = 1 - e^{-i\lambda s}.$$

(g) Applying the seasonal summation filter we obtain

$$S_t = Y_t + Y_{t-1} + \cdots + Y_{t-s+1}.$$

Differencing this we obtain

$$S_t - S_{t-1} = Y_t - Y_{t-s}.$$

(h) Let

$$\begin{aligned} A_{sd}(\lambda) &= \text{the transfer function for seasonal differencing,} \\ A_{ss}(\lambda) &= \text{the transfer function for seasonal summation,} \\ A_d(\lambda) &= \text{the transfer function for simpl differencing.} \end{aligned}$$

By the previous question

$$A_{sd}(\lambda) = A_d(\lambda) A_{ss}(\lambda),$$

and hence

$$A_{ss}(\lambda) = A_{sd}(\lambda)/A_d(\lambda).$$

Hence

$$A_{ss}(\lambda) = \frac{1 - e^{-i\lambda s}}{1 - e^{-i\lambda}}.$$

3. (a) After simple differencing the series still contains a strong seasonal pattern. After seasonal differencing, the series still contains a (linear) trend. This indicates that both seasonal and simple differencing are required to achieve stationarity.

   (b) The first statement estimates the autocorrelation function out to lag 24 months for the series obtained by differencing at lag 12 followed by differencing at lag 1. The second statement estimates the partial autocorrelation function out to lag 24 months for the series obtained by differencing at lag 12 followed by differencing at lag 1.

   (c) The pacf plot seems to show more evidence of long-term structure while the acf plot shows evidence of sharp cutoff. This suggests that an MA structure of some sort is probably appropriate. The acf plot shows one large correlation at lag 12 and one large correlation at lag 1. On the basis of this I would suggest fitting an ARIMA$(0, 1, 1) \times (0, 1, 1)_{12}$ model.

   (d) The model is:

$$(1 - L)(1 - L^{12})Y_t = (1 + \theta_1 L)(1 + \Theta_1 L^{12})\varepsilon_t.$$

   (e) Both parameters are significan so there is no redundancy, and the residual plots do not show significant correlation. On the basis of this we can say that there is no evidence to believe that the model is incorrect. (There does appear to be evidence of serial correlation in the raw residual plot, but it does not reach significance.

4. (a) Spectral estimates are obtained by averaging adjcent periodogram values. The bandwidth is the length of the frequency interval over which averaging takes place. The degrees of freedom is twice the number of values averaged. The distribution of the estimate is asymptotically a χ^2 random variable with this many degrees of freedom.

   (b) The power spectrum was obtained by averaging 5 adjacent periodogram values.

   (c) The model suggests that there is a periodic (seasonal) pattern to rainfall in Auckland. This variation must be very close to a pure cosinusoid because the spectrm shows a single high peak. A close look at the estimated spectrum shows there may be a small first harmonic, but it is very small.

   (d) Let $Y(t)$ indicate rainfall and $X(t)$ indicate temperature. The model is:

$$Y(t) = \sum_{u=-\infty}^{\infty} a(u)X(t - u) + \varepsilon(t)$$

(e) The relationship shows high coherence at the yearly and only relatively low coherence elsewhere. Outside of the yearly "peak" only about 5% of the coherence values are significance. This is what you would expect for unrelated phenomena.

(f) The relative phase of the yearly cycles is close to $-\pi$ (or 90°). This indicates that temperatures are high when rainfall is low and conversely. Note that there is a small deviation this though.