1. (a) The cases are as follows:

Case (i)	White noise.
Case (ii)	AR(1) with $\phi = 0.8$.
Case (iii)	MA(2) with positive coefficients.
Case (iv)	Possibly ARMA(1,2).
Case (v)	AR(12) with $\phi_1, \ldots, \phi_{11} = 0$ and $\phi_{12} = 0.6$.
Case (vi)	ARMA(1,0)$\times$(0,1).

Most of these would need confirmation by the pacf.

(b) The roots of the characteristic polynomial are 2 and $-2/3$. The series is not stationary and causal because one of these lies outside the unit disk.

(c) It is easy to show that for the MA(1) series $\rho(1) = \theta/(1+\theta^2)$. Now

$$0 \le (1+\theta)^2 = 1 + 2\theta + \theta^2 \quad \text{and} \quad 0 \le (1-\theta)^2 = 1 - 2\theta + \theta^2,$$

hence

$$-2\theta \le 1 + \theta^2 \quad \text{and} \quad 2\theta \le 1 + \theta^2$$

and so

$$-(1+\theta^2) \le 2\theta \le (1+\theta^2).$$

Dividing through by $2(1+\theta^2)$ yields

$$-\frac{1}{2} \le \frac{\theta}{(1+\theta^2)} \le \frac{1}{2}.$$

The kiddies will probably try calculus and fail. Any good attempt will get nearly full marks.

2. (a) The graph indicates that the variability present in the series increases with increasing mean level. This indicates that a variance stabilizing transformation is required. Also, it is probably proportional changes in demand which are important rather than absolute changes. Both these reasons indicate that log would be a good transformation to try. Since demand is a count, it might also be useful to consider square roots.

(b) The series is clearly non-stationary and the trend looks like it could be locally linear. This indicates that first differences could be appropriate.

(c) The trend has been removed by differencing once. The variability in the twice differenced series has been increased and estimates and forecasts would be of lower quality.

(d) Both the acf and the pacf seem to exhibit sharp cutoff after lag 1, but this is not possible with an ARMA model. There seems to be more structure present in the pacf so it looks like an AR(1) model might be the best choice.

(e) The first plot shows a single very large outlier in the last month of 1977. This seems to have produced runs of negative and positive values in its neighbourhood. Apart from that the plot looks ok. The Second plot shows a significant correlation at lag 12, indicating that there is a seasonal effect which is not being taken care of. The last plot shows the effect of the same correlation at lag 12.

(f) It looks like an additional term needs to be added. There is no indication at all of structure in the acf function so additional term will need to be in the MA part of the model. I would add a seaonal MA term at lag 12. I.e. move to a ARIMA$(0,1,1)\times(0,0,1)_{12}$ model.

3. (a) Filtering:

i. Let $\mathcal{A}$ indicate the filtering operation. The filter is linear if

$$\mathcal{A}[\alpha X + \beta Y](t) = \alpha\mathcal{A}[X](t) + \beta\mathcal{A}[Y](t).$$

and time invariant if

$$\mathcal{A}[L^u X](t) = L^u\mathcal{A}[X](t)$$

where L is the lag operator.

ii. Suppose the filter coefficents are $\{a(u)\}$, then

$$Y(t) = \sum_{u=-\infty}^{\infty} a(u)X(t-u).$$

iii. The transfer function is

$$A(\lambda) = \sum_{u=-\infty}^{\infty} a(u)e^{-i\lambda u}.$$

iv. The power spectrum is defined by

$$f_{XX}(\lambda) = \frac{1}{2\pi}\sum_{u=-\infty}^{\infty} c_{XX}(u)e^{-i\lambda u},$$

where $c_{XX}(u)$ is the autocovariance function of $X(t)$.

(b) i. The Fourier transform is defined to be

$$d_X^T(\lambda) = \sum_{t=0}^{T-1} X(t)e^{-i\lambda t}.$$

ii. Note that

$$\begin{aligned}\overline{e^{i\theta}} &= \overline{\cos\theta + i\sin\theta} \\ &= \cos\theta - i\sin\theta \\ &= e^{-i\theta}\end{aligned}$$

Thus

$$\begin{aligned}\overline{d_X^T(\lambda)} &= \overline{\sum_{t=0}^{T-1} X(t)e^{-i\lambda t}} \\ &= \sum_{t=0}^{T-1} X(t)\overline{e^{-i\lambda t}} \\ &= \sum_{t=0}^{T-1} X(t)e^{i\lambda t} \\ &= d_X^T(-\lambda).\end{aligned}$$

For any integer t, $e^{-i\lambda t}$ is 2π periodic. Thus $d_X^T(\lambda)$ is 2π periodic and hence the result.

iii. The asymptotic distribution of the discrete Fourier transform is given by:

$$\frac{d_X^T(\lambda)}{\sqrt{2\pi T}} \to N^c(0, f_{XX}(\lambda)$$

with distinct frequencies being asymptotically independent.

iv. The periodogram is defined by:

$$I_{XX}^T(\lambda) = \frac{1}{2\pi T}|d_X^T(\lambda)|^2.$$

v. The periodogram is not a consistent estimator since

$$\text{var } I_{XX}^T(\lambda) = f_{XX}(\lambda).$$

The estimator can be improved by smoothing. The simplest smoother is a simple moving average. If the number of values in the smoother tends to infinity as $T \to 0$, then the estimator will be consistent.

(c) i. The filter coefficents are

$$a(u) = \begin{cases} 1 & u = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

and the transfer function is

$$A(\lambda) = e^{i\lambda} - e^{-i\lambda} = 2i \sin \lambda.$$

The gain function is $2 \sin \lambda$ and the gain function is identically equal to $\pi/2$.

ii. The transfer function of the filter is

$$\begin{aligned} A(\lambda) &= \frac{e^{i\lambda}}{4} + \frac{1}{2} + \frac{e^{-i\lambda}}{4} \\ &= \frac{1 + \cos \lambda}{2} \end{aligned}$$

This is a smoothing filter because it is close to 1 for λ close to 0 and low when λ is close to π. However, it does pass at least 50% of all cosinusoids and so will pass quite a large amount of noise.

4. (a) i. The unit of time here is 100 years. Therefore x cycles per unit time translates into $x/100$ cycles per year.

ii. There is no natural origin for temperature series. Transformations like log or square root require a natural origin. (In the case of this particular series there are even negative values which would give log or square root indigestion.)

iii. We can work out the span from the given degrees of freedom (two df for each value in the span). In this case the spans were 3 and 5.

iv. Periodograms and spectra have variability proportional to mean level squared. Logarithms stabilize the variance. There is evidence of a periodicity at frequency .001. This corresponds to 1000 hundred year intervals or one cycle per 100,000 years. There is also evidence for a cycle at about 2.5 times for frequent – every 40,000 years. Both of these are apparent in the original series.

v. The series is unusual because the wave forms at the 100,000 year period are clearly not cosines, but there is no evidence of harmonics of this fundamental being present. It may be that they are being obscured by the presence of the 40,000 year cycle.

(b) i. There is evidence for a cycle with a frequency of roughly .01. This corresponds to a 100-year cycle. There is also weak evidence for a cycle with frequency between .06 and .07 which corresponds to a period between 14 and 17 years. There is no subspot cycle apparent.