1. (a) First note that:

$$\langle \mathbf{x}, \mathbf{1} \rangle = \sum_{i=1}^{n} x_i$$

and

$$\|\mathbf{1}\|^2 = \langle \mathbf{1}, \mathbf{1} \rangle = \sum_{i=1}^{n} 1 = n.$$

We can compute the projection of $\mathbf{x}$ onto $\mathcal{M}$ using the general formula.

$$\begin{aligned} P_{\mathcal{V}}\mathbf{x} &= \frac{\langle \mathbf{x}, \mathbf{1} \rangle}{\|\mathbf{1}\|^2}\mathbf{1} \\ &= \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \mathbf{1} \\ &= \overline{x}\mathbf{1} \\ &= (\overline{x}, \ldots, \overline{x}) \end{aligned}$$

Since $\mathbf{x} = P_{\mathcal{M}}\mathbf{x} + P_{\mathcal{V}}\mathbf{x}$,

$$\begin{aligned} P_{\mathcal{V}}\mathbf{x} &= \mathbf{x} - P_{\mathcal{M}}\mathbf{x} \\ &= (x_1, \ldots, x_n) - (\overline{x}, \ldots, \overline{x}) \\ &= (x_1 - \overline{x}, \ldots, x_n - \overline{x}) \end{aligned}$$

In words, $P_{\mathcal{V}}\mathbf{x}$ is the vector of deviations from the mean.

(b) Using the results above

$$\|P_{\mathcal{V}}\mathbf{x}\|^2 / \|\mathbf{1}\|^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})^2$$

Which is (very close to) the sample variance.

(c) The angle is defined by

$$\begin{aligned} \cos\theta &= \frac{\langle P_{\mathcal{V}}\mathbf{x}, P_{\mathcal{V}}\mathbf{y} \rangle}{\sqrt{\|P_{\mathcal{V}}\mathbf{x}\|^2 \|P_{\mathcal{V}}\mathbf{x}\|^2}} \\ &= \frac{\displaystyle\sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\displaystyle\sum_{i=1}^{n} (x_i - \overline{x})^2 \sum_{i=1}^{n} (y_i - \overline{y})^2}} \end{aligned}$$

which is the correlation between $\mathbf{x}$ and $\mathbf{y}$.

2. The following R commands will generate the plots.

   (a) `plot(ARMAacf(ar=c(1.2,-0.7),lag=15),type="h").`
   (b) `plot(ARMAacf(ar=c(-1,-0.6),lag=15),type="h").`
   (c) `plot(ARMAacf(ma=c(1.2, -0.7),lag=15),type="h").`
   (d) `plot(ARMAacf(ma=c(-1,-0.6),lag=15),type="h").`
   (e) `plot(ARMAacf(ar=0.7,ma=0.4,lag=15),type="h").`
   (f) `plot(ARMAacf(ar=0.7,ma=-0.4,lag=15),type="h").`

3. The important trigonometric formula here is

$$\cos\alpha\cos\beta = \frac{1}{2}\big[\cos(\alpha+\beta) + \cos(\alpha-\beta)\big].$$

To show weak stationarity we need to show that EY_t is constant and that $\text{cov}(Y_{t+u}, Y_t)$ depends only on u. Now

$$E(Y_t) = \frac{1}{2\pi}\int_0^{2\pi} a\cos(\lambda t+\theta)\, d\theta = 0$$

and

$$\begin{aligned}\text{cov}(Y_{t+u}, Y_t) &= \frac{1}{2\pi}\int_0^{2\pi} a^2\cos(\lambda(t+u)+\theta)\cos(\lambda t+\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2}{2}\big[\cos(\lambda(2t+u)+2\theta) + \cos(\lambda u)\big]\, d\theta \\ &= \frac{a^2\cos\lambda u}{2},\end{aligned}$$

which depends only on u and not t.

Showing strict stationarity is much harder. Let's just show that the distribution of Y_{t+u} does not depend on u. This means showing that

$$P[\cos(\lambda t+\theta) < y] = P[\cos(\lambda(t+u)+\theta) < y]$$

for any value of y. This in turn will follow if we can show

$$P[\cos(\theta) < y] = P[\cos(\theta + c) < y]$$

for every c, because both the probabilities above must be equal to $P[\cos(\theta) < y]$.