Mathematical details of the test statistics used by `kanova`

Rolf Turner

31st December 2024

1 Introduction

This note presents in some detail the formulae for the test statistics used by the `kanova()` function from the `kanova` package. These statistics are based on, and generalise, the ideas discussed in Diggle et al. (2000) and in Hahn (2012). They consist of sums of integrals (over the argument r of the K-function) of the usual sort of analysis of variance "regression" sums of squares, down-weighted over r by the estimated variance of the quantities being squared. The limits of integration r_0 and r_1 *could* be specified in the software (e.g. in the related `spatstat` function `studpermu.test()` they can be specified in the argument `rinterval`). However there is currently no provision for this in `kanova()`, and r_0 and r_1 are taken to be the min and max of the r component of the `"fv"` object returned by `Kest()`. Usually r_0 is 0 and r_1 is 1/4 of the length of the shorter side of the bounding box of the observation window in question.
There are test statistics for:

- one-way analysis of variance (one grouping factor),
- main effects in a two-way (two grouping factors) additive model, and
- a model with interaction versus an additive model in a two-way context.

2 The data

. In the context of a single classification factor A, with a levels, the data consist of K-functions $K_{ij}(r)$, $i = 1, \ldots, a$, $k = 1, \ldots, n_i$. The function $K_{ij}(r)$ is constructed (estimated) from an observed point pattern X_{ij}.
In the context of two classification factors A and B, with a levels and b levels respectively, the data consist of K-functions $K_{ijk}(r)$, $i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, n_{ij}$. The function $K_{ijk}(r)$ is constructed (estimated) from an observed point pattern X_{ijk}.
The observations have associated *weights*. The weight associated with $K_{ij}(r)$, in the single classification context, is $w_{ij} = m_{ij}^{\eta}$ where m_{ij} is the number of points in the pattern X_{ij} The exponent η is a constant that may be specified by the user of the `kanova` package. In the code η is denoted by `expo`, and defaults to 2.
In the context of two classification factors, the weight associated with $K_{ijk}(r)$ is $w_{ijk} = m_{ijk}^{\eta}$ where m_{ijk} is the number of points in the pattern X_{ijk}.
The test statistics used are calculated in terms of various weighted means of the observed K-functions. Explicitly we define

$$
\begin{aligned}
\tilde{K}_{i\bullet}(r) &= \frac{1}{w_{i\bullet}} \sum_{j=1}^{n_i} w_{ij} K_{ij}(r) \\
\tilde{K}_{\bullet\bullet}(r) &= \frac{1}{w_{\bullet\bullet}} \sum_{i=1}^{a} \sum_{j=1}^{n_i} w_{ij} K_{ij}(r) \\
&= \frac{1}{w_{\bullet\bullet}} \sum_{i=1}^{a} w_{i\bullet} \tilde{K}_{i\bullet}(r) \\
\tilde{K}_{ij\bullet}(r) &= \sum_{k=1}^{n_{ij}} \frac{w_{ijk}}{w_{ij\bullet}} K_{ijk}(r) \\
\tilde{K}_{i\bullet\bullet}(r) &= \sum_{j=1}^{b} \frac{w_{ij\bullet}}{w_{i\bullet\bullet}} \tilde{K}_{ij\bullet}(r) \\
&= \frac{1}{w_{i\bullet\bullet}} \sum_{j=1}^{b} \sum_{k=1}^{n_{ij}} w_{ijk} K_{ijk}(r) \\
\tilde{K}_{\bullet j\bullet}(r) &= \sum_{i=1}^{a} \frac{w_{ij\bullet}}{w_{\bullet j\bullet}} \tilde{K}_{ij\bullet}(r)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{w_{\bullet j \bullet}} \sum_{i=1}^{a} \sum_{k=1}^{n_{ij}} w_{ijk} K_{ijk}(r) \text{ and} \\
\tilde{K}_{\bullet\bullet\bullet}(r) &= \sum_{i=i}^{a} \frac{w_{i\bullet\bullet}}{w_{\bullet\bullet\bullet}} \tilde{K}_{i\bullet\bullet}(r) \\
&= \sum_{j=1}^{b} \frac{w_{\bullet j \bullet}}{w_{\bullet\bullet\bullet}} \tilde{K}_{\bullet j \bullet}(r) \\
&= \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{w_{ij\bullet}}{w_{\bullet\bullet\bullet}} \tilde{K}_{ij\bullet}(r) \\
&= \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n_{ij}} \frac{w_{ijk}}{w_{\bullet\bullet\bullet}} K_{ijk}(r)
\end{aligned}
$$

3 Variance functions

The variances of the K-functions are assumed to be proportional to functions which are constant over indices within each cell of the model. In the context of a single classification factor, the variance of $K_{ij}(r)$ is taken to be $\sigma_i^2(r)/w_{ij}$. It is assumed that under the null hypothesis of "no A effect", the functions $\sigma_i^2(r)$ are all equal to a single function, $\sigma^2(r)$. I.e. they do not vary with i. In the context of two classification factors, the variance of $K_{ijk}(r)$ is taken to be $\sigma_{ij}^2(r)/w_{ijk}$.
It is assumed that under the null hypothesis of "no A effect", the functions $\sigma_{ij}^2(r)$ do not vary with i, and are all equal to a single function $\sigma_j^2(r)$.

4 Estimating the variance functions

In the setting of a single classification factor, the variance function (unique under the null hypothesis), $\sigma^2(r)$ is estimated by

$$
s^2(r) = \frac{1}{n_{\bullet} - a} \sum_{i=1}^{a} \sum_{j=1}^{n_i} w_{ij} (K_{ij}(r) - \tilde{K}_{i\bullet}(r))^2 \ .
$$

Under the null hypothesis this is an unbiased estimate of $\sigma^2(r)$.

In the setting of two classification factors, where we are testing for an A effect, allowing for a B effect, the variance functions (depending only on the B effect under the null hypothesis), $\sigma_j^2(r)$) are estimated by

$$s_j^2(r) = \frac{1}{n_{\bullet j}} \sum_{i=1}^{a} \sum_{k=1}^{n_{ij}} w_{ijk}(K_{ijk}(r) - \tilde{K}_{ij\bullet}(r)) \ .$$

Under the null hypothesis these are a unbiased estimates of the $\sigma_j^2(r)$.
In the setting of two classification factors, where we are testing for interaction against an additive model (unlikely to arise as these circumstances may be) we need estimates of $\sigma_{ij}^2(r)$. These are given by

$$s_{ij}^2(r) = \frac{1}{n_{ij} - 1} \sum_{k=1}^{n_{ij}} w_{ijk}(K_{ijk}(r) - \tilde{K}_{ij\bullet})^2 \ .$$

These are a unbiased estimates of the $\sigma_{ij}^2(r)$.

5 The test statistics

In the setting of a single classification factor A, the statistic for testing for an A effect is

$$T = \sum_{i=1}^{a} n_i \int_{r_0}^{r_1} (\tilde{K}_i(r) - \tilde{K}(r))^2 / V_i(r) \, dr$$

where $V_i(r)$ is the estimated variance of $\tilde{K}_i(r) - \tilde{K}(r)$. This is given by

$$V_i(r) = s^2(r) \left(\frac{1}{w_{\ell\bullet}} - \frac{1}{w_{\bullet\bullet}} \right) \ .$$

In the setting of two classification factors A and B, the statistic for testing for an A effect allowing for a B effect is

$$T_A = \sum_{i=1}^{a} n_{i\bullet} \int_{r_0}^{r_1} (\tilde{K}_{i\bullet}(r) - \tilde{K}(r))^2 / V_{Ai}(r) \, dr$$

where $V_{Ai}(r)$ is the estimated variance of $\tilde{K}_{i\bullet}(r) - \tilde{K}(r)$. This is given by

$$V_{Ai}(r) = \tilde{s}_i^2(r) \left(\frac{1}{w_{i\bullet\bullet}} - \frac{2}{w_{\bullet\bullet\bullet}} \right) + \frac{1}{w_{\bullet\bullet\bullet}} \sum_{\ell=1}^{a} \frac{w_{i\bullet\bullet}}{w_{\bullet\bullet\bullet}} \tilde{s}_\ell^2(r) \ .$$

The foregoing expression may be re-written, more compactly, and in a form which makes it more obvious that the quantity is positive, as:

$$V_{Ai}(r) = \frac{1}{w_{\bullet\bullet}} \left[\sum_{\ell=1}^{a} \zeta_{i\ell} \times \tilde{s}_\ell^2(r) \right]$$

where

$$\begin{aligned}
\tilde{s}_\ell^2(r) &= \sum_{j=1}^{b} \frac{w_{\ell j\bullet}}{w_{\ell\bullet\bullet}} s_j^2(r),\ \ \ell = 1, \ldots, a, \\
\zeta_{i\ell} &= \begin{cases} \nu_\ell & \ell \neq i \\ \frac{(\nu_i - 1)^2}{\nu_i} & \ell = i \end{cases} \\
\nu_\ell &= \frac{w_{\ell\bullet\bullet}}{w_{\bullet\bullet\bullet}},\ \ \ell = 1, \ldots, a.
\end{aligned}$$

In the setting in which there are two classification factors and we are testing for interaction, against an additive models, the test statistic is

$$T_{AB} = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} \int_{r_0}^{r_1} (\tilde{K}_{ij\bullet}(r) - \tilde{K}_{i\bullet\bullet}(r) - \tilde{K}_{\bullet j\bullet}(r) + \tilde{K}(r))^2 / V_{ij}^{AB}(r)\, dr$$

where $V_{ij}^{AB}(r)$ is the (sample) variance of $\tilde{K}_{ij}(r) - \tilde{K}_{i\bullet}(r) - \tilde{K}_{\bullet j}(r) + \tilde{K}(r)$. The function $V_{ij}^{AB}(r)$ is even messier than $V_i^A(r)$. It is given by

$$\begin{aligned}
V_{ij}^{AB}(r) = s_{ij}^2(r) &\left(\frac{1}{w_{ij\bullet}} - \frac{2}{w_{i\bullet\bullet}} - \frac{2}{w_{\bullet j\bullet}} + \frac{2w_{ij\bullet}}{w_{\bullet j\bullet}} + \frac{2}{w_{\bullet\bullet\bullet}} \right) + \\
&\tilde{s}_{i\bullet}^2(r) \left(\frac{1}{w_{i\bullet\bullet}^2} - \frac{2}{w_{i\bullet\bullet} w_{\bullet\bullet\bullet}} \right) + \tilde{s}_{\bullet j}^2(r) \left(\frac{1}{w_{\bullet j\bullet}^2} - \frac{2}{w_{\bullet j\bullet} w_{\bullet\bullet\bullet}} \right) + \frac{\tilde{s}^2(r)}{w_{\bullet\bullet\bullet}}
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
\tilde{s}_{i\bullet}^2(r) &= \sum_{j=1}^{b} \frac{w_{ij\bullet}}{w_{i\bullet\bullet}} s_{ij}^2(r) \\
\tilde{s}_{\bullet j}^2(r) &= \sum_{i=1}^{a} \frac{w_{ij\bullet}}{w_{\bullet j\bullet}} s_{ij}^2(r) \text{ and} \\
\tilde{s}^2(r) &= \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{w_{ij\bullet}}{w_{\bullet\bullet\bullet}} s_{ij}^2(r)\ .
\end{aligned} \tag{2}$$

Here are some (terse) details about the variance of $\tilde{K}_{ij\bullet}(r) - \tilde{K}_{i\bullet\bullet}(r) - \tilde{K}_{\bullet j\bullet}(r) + \tilde{K}(r)$ as given by (1).

$$
\begin{aligned}
\mathrm{Var}(\tilde{K}_{ij\bullet}(r)) &= \frac{\sigma_{ij}^2(r)}{w_{ij\bullet}} \\
\mathrm{Var}(\tilde{K}_{i\bullet\bullet}(r)) &= \frac{\tilde{\sigma}_{i\bullet}^2(r)}{w_{i\bullet\bullet}} \\
\mathrm{Var}(\tilde{K}_{\bullet j\bullet}(r) &= \frac{\tilde{\sigma}_{\bullet j}^2(r)}{w_{\bullet j\bullet}} \\
\mathrm{Var}(\tilde{K}_{\bullet\bullet\bullet}(r)) &= \frac{\tilde{\sigma}^2(r)}{w_{\bullet\bullet\bullet}} \\
\mathrm{Cov}(\tilde{K}_{ij\bullet}(r), \tilde{K}_{i\bullet\bullet}) &= \frac{\sigma_{ij}^2(r)}{w_{i\bullet\bullet}} \\
\mathrm{Cov}(\tilde{K}_{ij\bullet}(r), \tilde{K}_{\bullet j\bullet}) &= \frac{\sigma_{ij}^2(r)}{w_{\bullet j\bullet}} \\
\mathrm{Cov}(\tilde{K}_{ij\bullet}(r), \tilde{K}_{\bullet\bullet\bullet}) &= \frac{\sigma_{ij}^2(r)}{w_{\bullet\bullet\bullet}} \\
\mathrm{Cov}(\tilde{K}_{i\bullet\bullet}(r), \tilde{K}_{\bullet j\bullet}) &= \frac{w_{ij\bullet}\sigma_{ij}^2(r)}{w_{i\bullet\bullet}w_{\bullet j\bullet}} \\
\mathrm{Cov}(\tilde{K}_{i\bullet\bullet}(r), \tilde{K}_{\bullet\bullet\bullet}) &= \frac{\tilde{\sigma}_{i\bullet}^2(r)}{w_{\bullet\bullet\bullet}w_{i\bullet\bullet}} \\
\mathrm{Cov}(\tilde{K}_{\bullet j\bullet}(r), \tilde{K}_{\bullet\bullet\bullet}(r) &= \frac{\tilde{\sigma}_{\bullet j}^2(r)}{w_{\bullet\bullet\bullet}w_{\bullet j\bullet}}
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\sigma}_{i\bullet}^2(r) &= \sum_{j=1}^{b} \frac{w_{ij\bullet}}{w_{i\bullet\bullet}} \sigma_{ij}^2(r) \\
\tilde{\sigma}_{\bullet j}^2(r) &= \sum_{i=1}^{a} \frac{w_{ij\bullet}}{w_{\bullet j\bullet}} \sigma_{ij}^2(r) \text{ and} \\
\tilde{\sigma}^2(r) &= \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{w_{ij\bullet}}{w_{\bullet\bullet\bullet}} \sigma_{ij}^2(r) \ .
\end{aligned}
$$

Note that the foregoing expression is just (2) with sample (estimated) quantities replaced by population quantities.

Sample calculation: to see that $\mathrm{Cov}(\tilde{K}_{ij\bullet}(r), \tilde{K}_{i\bullet\bullet}) = \sigma^2_{ij}/w_{i\bullet\bullet}$, note that $\tilde{K}_{i\bullet\bullet}(r)$ is a weighted sum over ℓ, of terms $\tilde{K}_{i\ell\bullet}(r)$.The K-functions involved correspond to independent patterns, and so are likewise independent. Consequently $\tilde{K}_{ij\bullet}(r)$ is independent of $\tilde{K}_{i\ell\bullet}(r)$, and the corresponding covariances are 0, except when $\ell = j$. We thus get only a single non-zero term from the sum of the covariances, explicitly

$$\mathrm{Cov}(\tilde{K}_{ij\bullet}(r), \frac{w_{ij\bullet}}{w_{i\bullet\bullet}}\tilde{K}_{ij\bullet}) = \frac{w_{ij\bullet}}{w_{i\bullet\bullet}}\mathrm{Var}(\tilde{K}_{ij\bullet}) = \frac{w_{ij\bullet}}{w_{i\bullet\bullet}}\frac{\sigma^2_{ij}}{w_{ij\bullet}} = \frac{\sigma^2_{ij}}{w_{i\bullet\bullet}} \; .$$

Finally we can obtain the variance term of interest, which is $\mathrm{Var}(\tilde{K}_{ij\bullet}(r) - \tilde{K}_{i\bullet\bullet}(r) - \tilde{K}_{j\bullet\bullet}(r) + \tilde{K}_{\bullet\bullet\bullet}(r))$. This expression is equal to

$$\begin{aligned}
&\mathrm{Var}(\tilde{K}_{ij\bullet}(r)) + \mathrm{Var}(\tilde{K}_{i\bullet\bullet}(r)) + \mathrm{Var}(\tilde{K}_{\bullet j\bullet}(r)) + \mathrm{Var}(\tilde{K}_{\bullet\bullet\bullet}(r)) \\
&- 2\mathrm{Cov}(\tilde{K}_{ij\bullet}(r), \tilde{K}_{i\bullet\bullet}(r)) - 2\mathrm{Cov}(\tilde{K}_{ij\bullet}(r), \tilde{K}_{\bullet j\bullet(r)}) + 2\mathrm{Cov}(\tilde{K}_{ij\bullet}(r), \tilde{K}_{\bullet\bullet\bullet}(r)) \\
&+ 2\mathrm{Cov}(\tilde{K}_{i\bullet\bullet}(r), \tilde{K}_{\bullet j\bullet}) - 2\mathrm{Cov}(\tilde{K}_{i\bullet\bullet}(r), \tilde{K}_{\bullet\bullet\bullet}(r)) \\
&- 2\mathrm{Cov}(\tilde{K}_{\bullet j\bullet}(r), \tilde{K}_{\bullet\bullet\bullet}(r)) \; .
\end{aligned}$$

Collecting terms in the foregoing expression, and using the previously stated symbolic representations of these terms, we obtain

$$\begin{aligned}
&\sigma^2_{ij}(r)\left(\frac{1}{w_{ij\bullet}} - \frac{2}{w_{i\bullet\bullet}} - \frac{2}{w_{\bullet j\bullet}} + \frac{2w_{ij\bullet}}{w_{i\bullet\bullet}w_{\bullet j\bullet}} + \frac{2}{w_{\bullet\bullet\bullet}}\right) + \\
&\tilde{\sigma}_{i\bullet}(r)\left(\frac{1}{w^2_{i\bullet\bullet}} - \frac{2}{w_{i\bullet\bullet}w_{\bullet\bullet\bullet}}\right) + \tilde{\sigma}_{\bullet j}(r)\left(\frac{1}{w^2_{\bullet j\bullet}} - \frac{2}{w_{\bullet j\bullet}w_{\bullet\bullet\bullet}}\right) + \frac{\tilde{\sigma}(r)}{w_{\bullet\bullet\bullet}} \; .
\end{aligned}$$

Replacing the population variances by their corresponding estimates (sample quantities) we obtain (1).

References

Peter J. Diggle, Jorge Mateu, and Helen E. Clough. A comparison between parametric and non-parametric approaches to the analysis of replicated spatial point patterns. *Advances in Applied Probability*, 32:331 – 343, 2000.

Ute Hahn. A studentized permutation test for the comparison of spatial point patterns. *Journal of the American Statistical Association*, 107(498): 754 – 764, 2012. DOI: 10.1080/01621459.2012.688463.